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Oscillation theorem for higher-order linear differential equations with periodic coefficients ☆

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ABSTRACT

In this paper, the zeros of solutions for higher-order linear differential equations with periodic coefficients are studied. It is shown that under certain hypotheses, the convergence exponent of zeros of the product of every fundamental solution is infinite.

1. Introduction and main results

In this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna values distribution theory of meromorphic functions [11,15]. In addition, we use the notation $\sigma(f)$ and $\lambda(f)$, respectively, to denote the order of growth and the exponent of convergence of the zeros of a meromorphic function f.

We define as in [7]

$$\sigma_e(f) = \overline{\lim_{r\to+\infty}} \frac{\log T(r,f)}{r}$$

to be the e-type order of a meromorphic function $f(z)$. Obviously, if $f(z)$ is entire, then

$$\sigma_e(f) = \overline{\lim_{r\to+\infty}} \frac{\log\log M(r,f)}{r}.$$

We also define as in [7]

$$\lambda_e(f) = \overline{\lim_{r\to+\infty}} \frac{\log N(r,f)}{r} \tag{1.1}$$

to be the e-type exponent of convergence of the zeros of $f(z)$.

Furthermore, we define the upper limit in (1.1) by $\lambda_{eR}(f)$ when we only count the zeros of $f(z)$ in the right half plane. Similarly, we define $\lambda_{eL}(f)$ to be the upper limit in (1.1) when we only count the zeros of $f(z)$ in the left half plane.

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For a set $E \subset (1, +\infty)$, we define

$$m(E) = \int_E dr, \qquad m_l(E) = \int_1^\infty \chi_E(t)\, dt/t,$$

where $\chi_E(t)$ denotes the characteristic function of the set E.

The study of the properties of solutions of a linear differential equation with periodic coefficients is one of the difficult aspects in the complex oscillation theory of differential equations. However, it is also one of the important aspects since it relates to many special functions. Many important researches were done by various authors, see, for instance, [1–9]. For the second-order periodic differential equation

$$f'' + A(z) f = 0, \tag{1.2}$$

S. Bank and I. Laine proved in [2]

Theorem 1.1. *Let $A(z)$ be a non-constant periodic entire function of period ω, which is of finite order of growth and transcendental in $e^{\alpha z}$, where $\alpha = 2\pi i \omega^{-1}$. If $f(z) \not\equiv 0$ is a solution of Eq.* (1.2) *with the property $\lambda(f) < \infty$, then $f(z)$ and $f(z+\omega)$ are linearly dependent.*

Y.M. Chiang and S.A. Gao proved the following theorem in [7].

Theorem 1.2. *Let $A(z) = B(e^z)$, where $B(\zeta) = g_1(1/\zeta) + g_2(\zeta)$, g_1 and g_2 are the entire functions with g_2 transcendental and $\sigma(g_2)$ not equal to a positive integer or infinity, and g_1 arbitrary.*

(i) *Suppose $\sigma(g_2) > 1$.*
  (a) *If f is a non-trivial solution of* (1.2) *with $\lambda_e(f) < \sigma(g_2)$, then $f(z)$ and $f(z+2\pi i)$ are linearly dependent.*
  (b) *If f_1 and f_2 are any two linearly independent solutions of* (1.2)*, then $\lambda_e(f_1 f_2) \geqslant \sigma(g_2)$.*

(ii) *Suppose $\sigma(g_2) < 1$.*
  (a) *If f is a non-trivial solutions of* (1.2) *with $\lambda_e(f) < 1$, then $f(z)$ and $f(z+2\pi i)$ are linearly dependent.*
  (b) *If f_1 and f_2 of* (1.2) *are any two linearly independent solutions, then $\lambda_e(f_1 f_2) \geqslant 1$.*

For second-order differential equation (1.2), if f_1 and f_2 are two linearly independent solutions, then

$$-4A = \frac{c^2}{E^2} - \frac{E'}{E}^2 + 2\frac{E''}{E},$$

where $E = f_1 f_2$. This formula plays an important role in the proofs of Theorem 1.1 and Theorem 1.2. But for higher-order differential equation, such formula does not exist. So that it is more difficult to investigate the properties of solutions for higher-order periodic differential equations.

For a higher-order periodic differential equation only with two terms, S.A. Gao proved the following result in [9].

Theorem 1.3. *Let $A(z) = B(e^z)$, where $B(\zeta) = g_1(\frac{1}{\zeta}) + g_2(\zeta)$, $g_1(t)$ and $g_2(t)$ are the entire functions, $g_1(t)$ (or $g_2(t)$) is transcendental and $\sigma(g_1)$ (or $\sigma(g_2)$) $< \frac{1}{2}$. If f is a non-trivial solution of the differential equation*

$$f^{(k)} + A(z) f = 0$$

with

$$\log^+ N(r, 1/f) = O(r),$$

then $f(z)$ and $f(z+2\pi i)$ are linearly dependent.

For a general higher-order periodic differential equation, Z.X. Chen, S.A. Gao and K.H. Shon proved the following theorem in [5].

Theorem 1.4. *Let A_j $(j = 0, \ldots, k-2)$ be entire functions of period $2\pi i$, $A_j(z) = C_j(\frac{1}{\zeta}) + B_j(\zeta)$, $\zeta = e^z$, and $C_j(t)$, $B_j(t)$ be entire functions with finite order of growth. Let $B_0(t)$ be transcendental with $\sigma(B_0) < \frac{1}{2}$, $\sigma(B_j) < \sigma(B_0)$ $(j = 1, \ldots, k-2)$ and $\sigma(C_s) < \sigma(B_0)$ $(s = 0, 1, \ldots, k-2)$ if $\sigma(B_0) > 0$; or let B_j $(j = 1, \ldots, k-2)$ and C_s $(s = 0, 1, \ldots, k-2)$ be polynomials if $\sigma(B_0) = 0$. If $f(z)$ is a non-trivial solution of*

$$f^{(k)} + A_{k-2} f^{(k-2)} + \cdots + A_0 f = 0, \tag{1.3}$$

and satisfies

$$\log^+ N(r, 1/f) = O(r),$$

then $f(z)$ and $f(z+2\pi i)$ are linearly dependent.

Comparing Theorem 1.2 and Theorem 1.3 (or Theorem 1.4), we can find that the condition in Theorem 1.3 (or Theorem 1.4) is stronger than that in Theorem 1.2. A natural question is what can be said for (1.3) where $A_0(z)$ still satisfies the hypotheses of Theorem 1.2. The main purpose of this paper is to study the problem. We will replace the condition "$A_0(z) = B_0(\zeta) + C_0(\frac{1}{\zeta})$, where B_0 is transcendental with $\sigma(B_0) < \frac{1}{2}$" in Theorem 1.4 with a weaker condition "$A_0(z) = G_0(\zeta) + g_0(\frac{1}{\zeta})$, where G_0 is transcendental with $\sigma(G_0)$ not equal to a positive integer or infinity." Our results (Theorem 1.5 and Theorem 1.6) are similar to Theorem 1.1 in some sense, while our methods are different from Theorem 1.2 and Theorem 1.3 (or Theorem 1.4).

Theorem 1.5. *Let $A_j(z) = B_j(e^z)$ $(j = 0, \ldots, k-2)$, where $B_j(\zeta) = G_j(\zeta) + g_j(1/\zeta)$, $G_j(\zeta)$ and $g_j(1/\zeta)$ are the entire functions. Let $G_0(\zeta)$ be transcendental with $\sigma(G_0) < +\infty$, $\sigma(G_j) < \sigma(G_0)$ $(j = 1, \ldots, k-2)$ and $\sigma(g_s) < \max\{\sigma(G_0), \sigma(g_0)\}$ $(s = 1, \ldots, k-2)$. Suppose that $f_1, \ldots, f_k$ are linearly independent solutions of Eq. (1.3) satisfying $\lambda_e(f_1 \cdots f_k) < \sigma(G_0)$. Suppose further that $f_j(z)$ and $f_j(z + q_j 2\pi i)$ $(j = 1, \ldots, k)$ are linearly dependent, where q_j $(1 \leqslant q_j \leqslant k,\ j = 1, \ldots, k)$ are some integers, then*

$$\sigma(G_0) = \frac{m}{q},$$

where q is the minimum common multiple of q_j $(j = 1, \ldots, k)$ and m is some integer.

Remark 1.1. The conclusion of Theorem 1.5 can be replaced by $\sigma(g_0) = m/q$, if G_j and g_j are transposed in the hypotheses above.

Theorem 1.6. *Let $A_j(z) = B_j(e^z)$ $(j = 0, \ldots, k-2)$, where $B_j(\zeta) = G_j(\zeta) + g_j(1/\zeta)$, G_j and g_j are the entire functions. Let $G_0(\zeta)$ be transcendental with $\sigma(G_0)$ not equal to a positive integer or infinity, $\sigma(G_j) < \sigma(G_0)$ $(j = 1, \ldots, k-2)$ and $\sigma(g_s) < \max\{\sigma(G_0), \sigma(g_0)\}$ $(s = 1, \ldots, k-2)$. Suppose that $f_1, \ldots, f_k$ are linearly independent solutions of Eq. (1.3) with $f_j(z)$ and $f_j(z+2\pi i)$ $(j = 1, \ldots, k)$ linearly dependent, then $\lambda_e(f_1 \cdots f_k) \geqslant \sigma(G_0)$, hence $\lambda(f_1 \cdots f_k) = +\infty$.*

Remark 1.2. The same conclusion remains valid if G_j and g_j are transposed in the hypotheses above.

2. Lemmas for the proof of theorems

Lemma 2.1. *(See [2].) Let $A(z)$ be entire, with period $2\pi i$, and such that*

$$\varlimsup_{r\to+\infty} \frac{\log\log M(r, A)}{r} = c < +\infty.$$

Then A has a representation

$$A(z) = A_1(e^z) + A_2(e^{-z}),$$

where A_1, A_2 are entire of order at most c. Also at least one of A_1, A_2 has order c.

Lemma 2.2. *(See [3].) Suppose that $k \geqslant 2$ and that $A_0, \ldots, A_{k-2}$ are the entire functions of period $2\pi i$, and that f is a non-trivial solution of (1.3). Suppose further that f satisfies $\log^+ N(r, 1/f) = o(r)$, that A_0 is non-constant and rational in e^z, and that if $k \geqslant 3$, then $A_1, \ldots, A_{k-2}$ are the constants. Then there exists an integer q with $1 \leqslant q \leqslant k$ such that $f(z)$ and $f(z + q2\pi i)$ are linearly dependent. The same conclusion holds if A_0 is transcendental in e^z, and f satisfies*

$$\log^+ N(r, 1/f) = O(r),$$

and if $k \geqslant 3$, then as $r \to +\infty$ through a set L_1 of infinite linear measure, we have

$$T(r, A_j) = o\big(T(r, A_0)\big)$$

for $j = 1, \ldots, k-2$.

Lemma 2.3. *(See [10].) Let ω be a transcendental meromorphic function with finite order $\sigma(\omega) = \rho < +\infty$. Let $\Gamma = \{(k_1, j_1), \ldots, (k_m, j_m)\}$ denote a finite set of distinct pairs of integers that satisfy $k_i > j_i \geqslant 0$ for $i = 1, \ldots, m$ and let $\varepsilon > 0$ be a given constant. Then there exists a set $E_1 \subset (1, +\infty)$ with $m_l(E_1) < +\infty$, such that for all z satisfying $|z| \notin E_1 \cup [0, 1]$ and $(k, j) \in \Gamma$, we have*

$$\left|\frac{\omega^{(k)}(z)}{\omega^{(j)}(z)}\right| \leqslant |z|^{(k-j)(\rho-1+\varepsilon)}.$$

Lemma 2.4. *Let $k \geqslant 2$, $B_j(\zeta) = G_j(\zeta) + g_j(1/\zeta)$ $(j = 0, \dots, k-2)$, where $G_j(\zeta)$ is transcendental and $\sigma(G_0) < +\infty$, $\sigma(G_j) < \sigma(G_0)$ $(j = 1, \dots, k-2)$, and $\sigma(g_s) < \max\{\sigma(G_0), \sigma(g_0)\}$ $(s = 1, \dots, k-2)$. Let $A_j(z) = B_j(e^z)$. Suppose that $f(z)$ $(\not\equiv 0)$ is a non-trivial solution of Eq.* (1.3) *satisfying $\log^+ N(r, 1/f) = O(r)$, then in $1 < |\xi| < \infty$, $f(z)$ can be represented as*

$$f(z) = \xi^d \psi(\xi) u(\xi) e^{h(\xi)},$$

where $\xi = e^{z/q}$, q $(1 \leqslant q \leqslant k)$ is an integer, d is some constant, $\psi(\xi)$ is analytic and does not vanish in $1 < |\xi| \leqslant \infty$ and $\psi(\infty) = 1$, both $u(\xi)$ and $h(\xi)$ are the entire functions of finite order. If G_j and g_j $(j = 0, \dots, k-2)$ are transposed in the hypotheses above, the same conclusion still holds with $\xi = e^{-z/q}$.

The proof of Lemma 2.4 is similar to the proof of Lemma 2.6 in [4]. We give the proofs for completeness here.

Proof of Lemma 2.4. By Lemma 2.1, we have

$$\sigma_e(A_j) < \sigma_e(A_0) \quad (j = 1, \dots, k-2),$$

so there exists a set $H \subset (0, +\infty)$ satisfying $m_l(H) = +\infty$, such that

$$T(r, A_j) = o\{T(r, A_0)\}, \quad r \in H. \tag{2.1}$$

Suppose that f $(\not\equiv 0)$ is a solution of (1.3) and satisfies $\log^+ N(r, 1/f) = O(r)$. By (2.1) and Lemma 2.2, we know that $f(z)$ and $f(z + q2\pi i)$ are linearly dependent for some integer q with $1 \leqslant q \leqslant k$. By [13, page 382], we can therefore write

$$f(z) = e^{d_1 z} G\bigl(e^{z/q}\bigr),$$

where d_1 is a constant, $G(\xi)$ is analytic in $0 < |\xi| < +\infty$, then [14, page 15] implies that in $1 < |\xi| < +\infty$,

$$G(\xi) = \xi^m \psi(\xi) u(\xi) e^{h(\xi)},$$

where m is an integer, $\psi(\xi)$ is analytic and does not vanish in $1 < |\xi|$ and $\psi(\infty) = 1$, $u(\xi)$ is the Weierstrass product formed with the zeros of G in $1 < |\xi| < +\infty$, $h(\xi)$ is an entire function. We assert that $u(\xi)$ and $h(\xi)$ are of finite order.

Firstly, we prove that $u(\xi)$ is of finite order of growth. Since for any $\rho > 1$, and any zero ξ_1 of G in $1 < |\xi| < \rho$, there exists at least one z_1 with $|z_1| < q(\log\rho + \pi)$ and $\exp(z_1/q) = \xi_1$ such that $f(z_1) = 0$, it follows that counting multiplicity, the number $N_1(\rho, 1/G)$ of zeros of G in $1 < |\xi| < \rho$ satisfies

$$\log^+ N_1(\rho, 1/G) \leqslant \log^+ N(r, 1/f) = O(r) = O(\log\rho).$$

This gives $\lambda(u(\xi)) < +\infty$, so $u(\xi)$ can be replaced by the canonical product and $u(\xi)$ is of finite order.

Secondly, we prove that $h(\xi)$ is of finite order of growth. Set $W(\xi) = \psi(\xi)u(\xi)$, then

$$f(z) = \xi^d W(\xi) e^{h(\xi)}, \tag{2.2}$$

where $d = d_1 q + m$. Substituting (2.2) into (1.3), yields

$$\bigl(h'\bigr)^k + P_{k-1}\bigl(h'\bigr) = 0, \tag{2.3}$$

where $P_{k-1}(h')$ is a differential polynomial in h' of total degree $k-1$, its coefficients are the polynomials in $\xi^m W^{(m)}/W$ $(m = 1, \dots, k)$, $\frac{1}{\xi}$ and $B_j(\xi^q)$ $(j = 0, \dots, k-2)$ with constant coefficients. Since $u(\xi)$ is of finite order and for $m = 1, \dots, k$, $\psi^{(m)}/\psi(\xi) = o(1)$ as $\xi \to +\infty$, by Lemma 2.4, there exists a subset $E_1 \subset (1, +\infty)$, with $m_l(E_1) < +\infty$ and a positive constant M, such that for $|\xi| \notin E_1 \cup [0, 1]$ and $m = 1, \dots, k$,

$$\bigl|\xi^m W^{(m)}/W\bigr| \leqslant |\xi|^M. \tag{2.4}$$

(Denote some fixed positive constant by M, M may be different at each occurrence.) Suppose φ is a meromorphic function in $1 < |\xi| < \infty$, define

$$m_1(\rho, \varphi) = \frac{1}{2\pi} \int_0^{2\pi} \log^+\bigl|\varphi\bigl(\rho e^{i\theta}\bigr)\bigr|\, d\theta \quad \bigl(|\xi| = \rho\bigr).$$

Then, by (2.4), we have $m_1(\rho, \xi^m \frac{W^{(m)}}{W}) = O(\log\rho)$. Using the same argument as in the proof of the Clunie Lemma, noting the fact that $g_j(1/\xi^q) = O(1)$ $(j = 0, \dots, k-2)$ as $\xi \to \infty$, by (2.3), (2.4), we can obtain

$$m(\rho, h') = O\left\{\sum_{j=0}^{k-2} m(\rho, G_j(\xi^q)) + \log m(\rho, h') + \log\rho\right\}$$
$$\leqslant M\{m(\rho, G_0(\xi^q)) + \log m(\rho, h') + \log\rho\}, \tag{2.5}$$

where $\rho \notin E_2$, $E_2 \subset (0, +\infty)$ with $m(E_2) < +\infty$, M $(M > 0)$ is a constant. Because $G_0(\zeta)$ is of finite order, by (2.5), we see that h' is of finite order, hence h is of finite order. □

Lemma 2.5. *(See [12].) Let $w(z)$ be an entire transcendental function, then for any $\varepsilon > 0$,*

$$\nu(r)^{1-\varepsilon} < \log\mu(r)$$

holds except for a set in $[1, \infty)$ with finite logarithmic measure, where $\nu(r)$ and $\mu(r)$ are the central index and maximum term of $w(z)$ respectively.

Lemma 2.6. *Suppose $f(z)$ is an entire function satisfying*

$$f(z) = \xi^d G(\xi),$$

where d is a constant, $\xi = e^{z/q}$, q is an integer, $G(\xi)$ is analytic in $0 < |\xi| < +\infty$, then

$$\lambda_e(f) = \frac{1}{q}\varlimsup_{\rho\to\infty}\frac{\log n(\rho^{-1} \leqslant |\xi| \leqslant \rho, 1/G(\xi))}{\log\rho}$$
$$= \frac{1}{q}\max\{\lambda_0(G), \lambda_\infty(G)\},$$

and $\lambda_{eR}(f) = \frac{1}{q}\lambda_\infty(G)$, $\lambda_{eL}(f) = \frac{1}{q}\lambda_0(G)$.

The idea of proof of Lemma 2.6 is similar to the idea in [7, page 277].

Proof. Since the transformation $\xi = e^{z/q}$ is one–one correspondence between the sets $\{z\colon -q\log\rho \leqslant Re\, z \leqslant q\log\rho,\ -q\pi \leqslant Im\, z \leqslant q\pi\}$ and $\{\xi\colon\ \rho^{-1} \leqslant |\xi| \leqslant \rho\}$. By the periodicity of $e^{z/q}$, we have

$$\begin{aligned}
n\left(\rho^{-1} \leqslant |\xi| \leqslant \rho, \frac{1}{G(\xi)}\right) &= n\left(\left\{\begin{array}{c} -q\log\rho \leqslant Re\, z \leqslant q\log\rho \\ -q\pi \leqslant Im\, z \leqslant q\pi \end{array}\right\}, \frac{1}{f(z)}\right) \\
&\leqslant n\left(|z| \leqslant q\log\rho + q\pi, \frac{1}{f(z)}\right) \\
&\leqslant \frac{\log\rho + \pi}{\pi} \times n\left(\left\{\begin{array}{c} -(q\log\rho + q\pi) \leqslant Re\, z \leqslant q\log\rho + q\pi \\ -q\pi \leqslant Im\, z \leqslant q\pi \end{array}\right\}, \frac{1}{f(z)}\right) \\
&= \frac{\log\rho + \pi}{\pi} \times n\left(\left(e^\pi\rho\right)^{-1} \leqslant |\xi| \leqslant e^\pi\rho, \frac{1}{G(\xi)}\right).
\end{aligned}$$

Thus

$$\lambda_e(f(z)) = \frac{1}{q}\varlimsup_{\rho\to+\infty}\frac{\log n(\rho^{-1} \leqslant |\xi| \leqslant \rho, \frac{1}{G(\xi)})}{\log\rho}.$$

Similarly, we have

$$\lambda_{eR}(f) = \frac{1}{q}\varlimsup_{\rho\to\infty}\frac{\log n(1 \leqslant |\xi| \leqslant \rho, \frac{1}{G(\xi)})}{\log\rho} = \frac{1}{q}\lambda_\infty(G),$$
$$\lambda_{eL}(f) = \frac{1}{q}\varlimsup_{\rho\to\infty}\frac{\log n(\rho^{-1} \leqslant |\xi| \leqslant 1, \frac{1}{G(\xi)})}{\log\rho} = \frac{1}{q}\lambda_0(G).$$

Thus,

$$\lambda_e(f) = \frac{1}{q}\max\{\lambda_0(G), \lambda_\infty(G)\}. \qquad \square$$

Lemma 2.7. *(See [14].) Let $f(z)$ be an entire function of finite order ρ, $a_1, a_2, \ldots,$ be the non-zero zeros of $f(z)$, then there exist a constant $M > 0$, h $(> \rho)$ and an R-set U consisting of a countable union of discs $B(a_n, |a_n|^{-h}) = \{z\colon\ |z - a_n| < |a_n|^{-h}\ (n = 1, 2, \ldots)\}$ such that for an integer j and for all large z not in U, we have*

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leqslant |z|^M.$$

3. Proof of Theorem 1.5

Proof. Since $\lambda_e(f_1 \cdots f_k) < \sigma(G_0) < \infty$, we have $\lambda_e(f_j) < \sigma(G_0) < \infty$ $(j = 1, \ldots, k)$, then by Lemma 2.4, $f_j(z)$ has the representation in $1 < |\xi_j| < \infty$,

$$f_j(z) = \xi_j^{b_j} \psi(\xi_j) u(\xi_j) e^{h(\xi_j)}, \tag{3.1}$$

where $\xi_j = e^{z/q_j}$, q_j $(1 \leqslant q_j \leqslant k,\ \ j = 1, \ldots, k)$ are some integers, b_j are some constants. Let q be the minimum common multiple of $q_1, \ldots, q_k$ and let $d_j = q/q_j$ $(j = 1, \ldots, k)$, then from (3.1), we have

$$\begin{aligned} f_j(z) &= \xi^{c_j} \psi\big(\xi^{d_j}\big) u\big(\xi^{d_j}\big) e^{h(\xi^{d_j})} \\ &= \xi^{c_j} \psi_j(\xi) u_j(\xi) e^{h_j(\xi)} \\ &= \xi^{c_j} \omega_j(\xi) e^{h_j(\xi)}, \end{aligned} \tag{3.2}$$

for $j = 1, \ldots, k$, where $\xi = e^{z/q}$, c_j are some constants, $\psi_j(\xi)$ are analytic in $1 < |\xi|$ (including ∞) and $\psi_j(\infty) \neq 0$, $u_j(\xi)$ and $h_j(\xi)$ are the entire functions of finite order. Set $E = f_1 \cdots f_k$, then from (3.2),

$$\begin{aligned} \Phi(\xi) = E(z) &= \xi^{\sum_1^k c_j} \prod_1^k \psi_j(\xi) \prod_1^k u_j(\xi) e^{\sum_1^k h_j(\xi)} \\ &= \xi^{\sum_1^k c_j} \psi(\xi) u(\xi) e^{h(\xi)} \\ &= \xi^{\sum_1^k c_j} H(\xi) e^{h(\xi)}, \end{aligned} \tag{3.3}$$

where $\psi(\xi)$ is analytic on $1 < |\xi|$ (including ∞), $\psi(\infty) \neq 0$, $u(\xi)$ and $h(\xi)$ are the entire functions of finite order. We assert that $h(\xi)$ is a polynomial. In fact, from (1.3), we know that the Wronskian $w(f_1 \cdots f_k)$ is a non-zero constant, say c, then we can write

$$\frac{c}{E} = \frac{w(f_1 \cdots f_k)}{f_1 \cdots f_k},$$

thus $1/E$ is represented as a determinant in the functions $f_j^{(m)}/f_j$ $(j = 1, \ldots, k,\ \ m = 1, \ldots, k-1)$. On the other hand, if we set $l_j(\xi) = f_j'(z)/f_j(z)$, then from (3.2), we have $\sigma_\infty(l_j(\xi)) < \infty$. Combining this fact and (3.3), we know that $h(\xi)$ is a polynomial. Suppose the degree of $h(\xi)$ is m. From (3.3) and Lemma 2.6, we have

$$\lambda_{eR}\big(E(z)\big) = \frac{1}{q}\lambda(u),$$

this gives

$$\lambda(u) = q\lambda_{eR}\big(E(z)\big) = \tau \leqslant q\lambda_e\big(E(z)\big) < q\sigma(G_0) = \sigma\big(D(\xi)\big) = \sigma, \tag{3.4}$$

where $D(\xi) = G_0(\zeta) = G_0(\xi^q)$. We take an ε (> 0) such that

$$\sigma(1 - \varepsilon) > \tau + \varepsilon. \tag{3.5}$$

Let $\nu(\rho)$ and $\mu(\rho)$ be the central index and the maximum term of $D(\xi)$, then

$$\varlimsup_{\rho \to +\infty} \frac{\log \nu(\rho)}{\log \rho} = \sigma. \tag{3.6}$$

By the Cauchy inequality, $\mu(\rho) \leqslant M(\rho, D(\xi))$ holds, combining this and Lemma 2.5, there exists a set E_0 with $m_l(E_0) < +\infty$ such that for $|\xi| = \rho \in [1, \infty) \setminus E_0$,

$$\nu(\rho)^{1-\varepsilon} < \log M\big(\rho, D(\xi)\big). \tag{3.7}$$

On the other hand, from the Wiman–Valiron theory, there exists a set $E_1 \subset (1, +\infty)$ with $m_l(E_1) < +\infty$ such that for ξ satisfying $|\xi| = \rho \notin [0, 1] \cup E_1$ and $|D(\xi)| = M(\rho, D(\xi))$, we have

$$\frac{D'(\xi)}{D(\xi)}=\big(1+o(1)\big)\frac{\nu(\rho)}{\xi}. \tag{3.8}$$

In (3.2), since

$$\frac{\psi_j^{(m)}(\xi)}{\psi_j(\xi)}=o(1)\quad (m=1,\dots,k) \tag{3.9}$$

as $\xi\to\infty$ and $u_j(\xi)$ and $h_j(\xi)$ are the entire functions of finite order, by Lemma 2.3, there exist a constant $M>0$, $\lambda>0$, $\rho_0>0$ and an R-set U consisting of a countable union of discs $B(\eta_\nu,|\eta_\nu|^{-\lambda})=\{\xi\colon\ |\xi-\eta_\nu|<|\eta_\nu|^{-\lambda}\}$ such that for $|\xi|>\rho_0$ and $\xi\notin U$,

$$\left|\frac{\omega_j^{(m)}(\xi)}{\omega_j(\xi)}\right|+\left|\frac{h_j^{(m)}(\xi)}{h_j(\xi)}\right|+\left|\frac{D'(\xi)}{D(\xi)}\right|\leqslant|\xi|^M. \tag{3.10}$$

Let V be the R-set consisting of discs $\{\xi\colon\ |\xi-\eta_\nu|<2|\eta_\nu|^{-\lambda}\}$. By Lemma 2.3 and (3.9), there exists a set $E_2\subset(1,\infty)$ with $m_l(E_2)<+\infty$ such that $\xi\notin V$ and

$$\left|\frac{H'(\xi)}{H(\xi)}\right|\leqslant|\xi|^{\tau+\varepsilon-1}, \tag{3.11}$$

for $|\xi|\notin E_2$. From (3.6), we assert that there exists a set $E\subset(1,+\infty)$ with $m_l(E)=+\infty$ such that

$$\lim_{\substack{\rho\to+\infty\\ \rho\in E}}\frac{\log\nu(\rho)}{\log\rho}=\sigma. \tag{3.12}$$

In fact, by (3.6), there exists a sequence $\{\rho_n\}$ $(\rho_n\to\infty)$ such that

$$\lim_{n\to\infty}\frac{\log\nu(\rho_n)}{\log\rho_n}=\sigma,$$

set $E=\bigcup_{n=1}^{\infty}[\rho_n,\rho_n+1]$, then $m_l(E)=+\infty$ and (3.12) holds. Thus we can take a $\rho_n\in E\setminus(E_0\cup E_1\cup E_2)$, for sufficiently large n, from (3.12), that

$$\rho_n^{\sigma(1-\varepsilon)}<\nu(\rho_n)<\rho_n^{\sigma(1+\varepsilon)} \tag{3.13}$$

holds. It follows from (3.7), (3.13) that for sufficiently large n,

$$\log M\big(\rho_n,D(\xi)\big)>\rho_n^{\sigma(1-\varepsilon)^2}. \tag{3.14}$$

Now, we assert that there exists a constant $d>0$, such that for ξ_n satisfying $|\xi_n|=\rho_n$ and $|D(\xi_n)|=M(\rho_n,D(\xi))$ and for a sufficiently large n,

$$\log\big|D(\xi)\big|>|\xi|^{\sigma(1-\varepsilon)^3} \tag{3.15}$$

holds in the disc $D_n=\{\xi\colon\ |\xi-\xi_n|<|\xi_n|^{-d}\}$. In order to explain the existence of d, we have a discussion as follows.

Firstly, we can choose a constant d (>0) such that D_n and the R-set U are disjoint. In fact, since the circle $|\xi|=\rho_n$ and the R-set V are disjoint, the circle $|\xi|=\rho_n$ and the R-set U must be disjoint. We will divide our discussion into two cases:

Case (i). η_ν lies in the circle $|\xi|=\rho_n$. Then $|\eta_\nu|<|\xi_n|=\rho_n$. Since $|\xi|=\rho_n$ and the R-set V are disjoint, $|\eta_\nu|+2|\eta_\nu|^{-\lambda}<|\xi_n|=\rho_n$ holds. We use "dis" to denote the distance, then

$$\begin{aligned}\operatorname{dis}\big(|\xi-\xi_n|<|\xi_n|^{-d},|\xi-\eta_\nu|<|\eta_\nu|^{-\lambda}\big)&=|\xi_n-\eta_\nu|-|\xi_n|^{-d}-|\eta_\nu|^{-\lambda}\\&\geqslant|\xi_n|-|\eta_\nu|-|\xi_n|^{-d}-|\eta_\nu|^{-\lambda}\\&>2|\eta_\nu|^{-\lambda}-|\xi_n|^{-d}-|\eta_\nu|^{-\lambda}\\&=|\eta_\nu|^{-\lambda}-|\xi_n|^{-d}.\end{aligned}$$

We take a d such that $d>\lambda$, then D_n and $\{\xi\colon\ |\xi-\eta_\nu|<|\eta_\nu|^{-\lambda}\}$ are disjoint.

Case (ii). η_ν lies outside of the circle $|\xi|=\rho_n$. Then $|\eta_\nu|>|\xi_n|=\rho_n$, and $|\eta_\nu|-2|\eta_\nu|^{-\lambda}>|\xi_n|=\rho_n$. We divide this case into two subcases.

Subcase (a). If $|\eta_\nu|<|\xi_n|^2$, then

$$\begin{aligned}\operatorname{dis}\big(|\xi-\xi_n|<|\xi_n|^{-d},|\xi-\eta_\nu|<|\eta_\nu|^{-\lambda}\big)&=|\xi_n-\eta_\nu|-|\xi_n|^{-d}-|\eta_\nu|^{-\lambda}\\&>2|\eta_\nu|^{-\lambda}-|\xi_n|^{-d}-|\eta_\nu|^{-\lambda}\\&=|\eta_\nu|^{-\lambda}-|\xi_n|^{-d}.\end{aligned}$$

We can choose d such that $d>2\lambda$, then D_n and $\{\xi\colon\ |\xi-\eta_\nu|<|\eta_\nu|^{-\lambda}\}$ are disjoint.

Subcase (b). If $|\eta_\nu| \geqslant |\xi_n|^2$, then

$$\begin{aligned}\operatorname{dis}\big(|\xi-\xi_n|<|\xi_n|^{-d}, |\xi-\eta_\nu|<|\eta_\nu|^{-\lambda}\big) &= |\xi_n-\eta_\nu| - |\xi_n|^{-d} - |\eta_\nu|^{-\lambda} \\ &\geqslant |\xi_n|^2 - |\xi_n| - |\xi_n|^{-d} - |\eta_\nu|^{-d} \to +\infty \quad (n\to+\infty),\end{aligned}$$

D_n and $\{\xi\colon |\xi-\eta_\nu|<|\eta_\nu|^{-\lambda}\}$ are disjoint for any $d>0$. From case (i)(ii), we can choose a d $(d>2\lambda)$ such that D_n and the R-set U are disjoint for sufficiently large n.

Secondly, we will choose the d further such that (3.15) holds in D_n for sufficiently large n. For any $\xi \in D_n$,

$$\log D(\xi) = \log D(\xi_n) + \int_{\xi_n}^{\xi} \frac{D'(t)}{D(t)}\,dt, \tag{3.16}$$

where the integration path is the line segment connecting ξ with ξ_n. Taking the real parts of both sides of (3.16), we have

$$\log\big|D(\xi)\big| = \log\big|D(\xi_n)\big| + Re\int_{\xi_n}^{\xi} \frac{D'(t)}{D(t)}\,dt. \tag{3.17}$$

From (3.10), (3.14), (3.17) and the fact that $\frac{1}{2}\rho_n < |\xi| < 2\rho_n$ for sufficiently large n, we have

$$\begin{aligned}\log\big|D(\xi)\big| &\geqslant \log\big|D(\xi_n)\big| - \left|\int_{\xi_n}^{\xi} \frac{D'(t)}{D(t)}\,dt\right| \\ &\geqslant \log\big|D(\xi_n)\big| - |\xi-\xi_n|\big(|\xi_n|+|\xi_n|^{-d}\big)^M \\ &\geqslant \log\big|D(\xi_n)\big| - |\xi_n|^{-d}\big(|\xi_n|+|\xi_n|^{-d}\big)^M \\ &> |\xi_n|^{\sigma(1-\varepsilon)^2} - |\xi_n|^{-d}\big(|\xi_n|+|\xi_n|^{-d}\big)^M.\end{aligned}$$

We can choose the d such that $d>\max\{2\lambda, M\}$, then (3.15) holds in D_n for sufficiently large n.

Substituting $f_j(z) = \xi^{c_j}\omega_j(\xi)e^{h_j(\xi)}$ into (1.3), we have

$$\big(h_j'\big)^k + P_{k-1}(\xi)\big(h_j'\big)^{k-1} + \sum_{i=0}^{k-2} P_i(\xi)\big(h_j'\big)^i + \frac{q^k}{\xi^k}\left[g_0\left(\frac{1}{\xi^q}\right) + G_0\big(\xi^q\big)\right] = 0, \tag{3.18}$$

where $P_{k-1}(\xi)$ is a polynomial in $\frac{\omega_j^{(m)}(\xi)}{\omega_j(\xi)}$, $\frac{h_j^{(m)}(\xi)}{h_j'(\xi)}$, $\frac{1}{\xi^s}$ $(1\leqslant s\leqslant k-1,\ 1\leqslant m\leqslant k)$ with constant coefficients, $P_i(\xi)$ $(i=0,\dots,k-2)$ are the polynomials in $\frac{\omega_j^{(m)}(\xi)}{\omega_j(\xi)}$, $\frac{h_j^{(m)}(\xi)}{h_j'(\xi)}$, $\frac{1}{\xi^s}$ $(1\leqslant s\leqslant k-1,\ 1\leqslant m\leqslant k)$ and $g_1(\frac{1}{\xi^q}),\dots,g_{k-2}(\frac{1}{\xi^q})$, $G_1(\xi^q),\dots,G_{k-2}(\xi^q)$ with constant coefficients. By (3.10), we have

$$\big|P_{k-1}(\xi)\big| \leqslant |\xi|^M. \tag{3.19}$$

We take τ_1 such that for $j=1,\dots,k-2$,

$$\sigma\big(G_j\big(\xi^q\big)\big) = q\sigma\big(G_j(\zeta)\big) < \tau_1 < q\sigma\big(G_0(\zeta)\big) = \sigma\big(G_0\big(\xi^q\big)\big) = \sigma,$$

then $G_j(\xi^q)$ satisfies for sufficiently large ρ,

$$\log\big|G_j\big(\xi^q\big)\big| < \rho^{\tau_1}.$$

So when $\xi\in D_n$, $n\to\infty$, for $j=1,\dots,k-2$, by (3.15), we get

$$\begin{aligned}&\log\big|G_j\big(\xi^q\big)\big| < \rho^{\tau_1} < \rho^{\sigma(1-\varepsilon)^3} < \log\big|G_0\big(\xi^q\big)\big| \\ &\quad\left(|\xi|=\rho,\ 0<\varepsilon<1-\left(\frac{\tau_1}{\sigma}\right)^{1/3}\right).\end{aligned} \tag{3.20}$$

Because $g_j(\frac{1}{\xi^q}) = o(1)$ $(j=0,\dots,k-2)$ as $\xi\in D_n, n\to\infty$, by (3.10), (3.18), (3.20), we have for $i=0,\dots,k-2$,

$$\log\big|P_i(\xi)\big| < \rho^{\tau_1} < \rho^{\sigma(1-\varepsilon)^3} < \log\big|G_0\big(\xi^q\big)\big| \quad \big(|\xi|=\rho\big). \tag{3.21}$$

Set $F(\xi)=\frac{q^k}{\xi^k}[g_0(\frac{1}{\xi^q})+G_0(\xi^q)]$, then $\sigma(F)=\sigma(G_0(\xi^q))=\sigma(D)=\sigma$. Because when $n\to\infty$, we have $\xi_n\to\infty$ and $\xi\to\infty$, $g_0(\frac{1}{\xi^q})=o(1)$, by (3.15), (3.21), we have

$$\log\left|P_i(\xi)\right|<\rho^{\tau_1}<\rho^{\sigma(1-\varepsilon)^3}<\log\left|F(\xi)\right|\quad(i=0,\dots,k-1)\tag{3.22}$$

for $\xi\in D_n$ and sufficiently large n.

Now we estimate h_j' on $D_n=\{\xi\colon |\xi-\xi_n|<|\xi_n|^{-d}\}$. We define a single valued branch of $F(\xi)^{\frac{1}{k}}$. By (3.18), we have

$$\left(\frac{h_j'}{F^{\frac{1}{k}}}\right)^k+\left(\frac{P_{k-1}}{F^{\frac{1}{k}}}\right)\left(\frac{h_j'}{F^{\frac{1}{k}}}\right)^{k-1}+\sum_{i=0}^{k-2}\left(\frac{P_i}{F^{\frac{k-i}{k}}}\right)\left(\frac{h_j'}{F^{\frac{1}{k}}}\right)^i+1=0.\tag{3.23}$$

By (3.22), on D_n, for n sufficiently large, all points ξ satisfy

$$\frac{P_i(\xi)}{F^{\frac{1}{k}}}\to 0\quad(\xi\to\infty,\ i=0,\dots,k-1).\tag{3.24}$$

If $\frac{h_j'}{F^{\frac{1}{k}}}$ is unbounded on D_n, then there exist infinitely many n, say n_j, such that on D_{n_j}, there is a point ξ_{n_j} satisfying as $n_j\to\infty$,

$$\frac{h_j'(\xi_{n_j})}{[F(\xi_{n_j})]^{\frac{1}{k}}}\to\infty.\tag{3.25}$$

By (3.24), (3.25), we see (3.23) is a contradiction. So, $\frac{h_j'}{F^{\frac{1}{k}}}$ is bounded on D_n. By (3.23) and (3.24), we see that on D_n, as $n\to\infty$,

$$\frac{h_j'(\xi)}{F^{\frac{1}{k}}(\xi)}\to c,\tag{3.26}$$

uniformly, where $c^k=-1$. By (3.24), on D_n, (3.18) can be written as

$$\left[h_j'(\xi)\right]^k\left\{1+\frac{P_{k-1}(\xi)}{h_j'(\xi)}+\sum_{i=0}^{k-2}\frac{P_i(\xi)}{[h_j'(\xi)]^{k-i}}\right\}+F(\xi)=0.\tag{3.27}$$

By (3.19), (3.22), (3.24), (3.26), we have

$$\frac{P_{k-1}(\xi)}{h_j'(\xi)}=\frac{O(\rho^M)}{h_j'(\xi)},\qquad\sum_{i=0}^{k-2}\frac{P_i(\xi)}{(h_j'(\xi))^{k-i}}=\frac{o(1)}{h_j'(\xi)}.\tag{3.28}$$

So, (3.27), (3.28) give that

$$h_j'(\xi)\left[1+\frac{O(\rho^M)}{h_j'(\xi)}\right]=c_{j,n}F^{\frac{1}{k}}(\xi),\tag{3.29}$$

where $c_{j,n}^k=-1$. Thus on D_n, $h_j'-c_{j,n}F^{\frac{1}{k}}=O(\rho^M)$, and there exists a constant M such that on D_n, for every sufficiently large n,

$$\left|h_j'(\xi)-c_{j,n}F^{\frac{1}{k}}(\xi)\right|\leqslant\rho^M.\tag{3.30}$$

Recall $F(\xi)=\frac{q^k}{\xi^k}[g_0(\frac{1}{\xi^q})+G_0(\xi^q)]$, (3.30) can be written as

$$\left|h_j'(\xi)-c_{j,n}\frac{q}{\xi}D^{\frac{1}{k}}(\xi)\right|=O\left(\rho^M\right).\tag{3.30$'$}$$

On the other hand, from (3.2), we have for $\xi\in D_n$, $f_j(z)=\xi^{c_j}\omega_j(\xi)e^{h_j(\xi)}$, set $W_j(\xi)=\omega_j(\xi)e^{h_j(\xi)-Q_j(\xi)}$, where

$$Q_j(\xi)=\int_{\xi_n}^{\xi}\frac{qc_{j,n}D(t)^{1/k}}{t}\,dt.\tag{3.31}$$

Thus for $\xi\in D_n$,

$$f_j(z) = \xi^{c_j}\omega_j(\xi)e^{h_j(\xi)} = \xi^{c_j}W_j(\xi)e^{Q_j(\xi)},$$

this gives

$$\frac{f_j'(z)}{f_j(z)} = \frac{1}{q}\left(c_j + \xi\frac{\omega_j'}{\omega_j} + \xi h_j'\right) = \frac{1}{q}\left(c_j + \xi\frac{W_j'}{W_j} + \xi Q_j'\right),$$

so,

$$\frac{W_j'}{W_j} = \frac{\omega_j'}{\omega_j} + h_j' - \frac{qc_{j,n}D(\xi)^{1/k}}{\xi}. \tag{3.32}$$

By (3.10), (3.30′), (3.32), there exists a constant $M > 0$ such that for $\xi \in D_n$ and sufficiently large n,

$$\left|\frac{W_j'(\xi)}{W_j(\xi)}\right| \leqslant |\xi|^M. \tag{3.33}$$

Since $\frac{W_j^{(m)}(\xi)}{W_j(\xi)}$ is a differential polynomial in $\frac{W_j'(\xi)}{W_j(\xi)}$ with constant coefficients and

$$\left(\frac{W_j'(\xi_n)}{W_j(\xi_n)}\right)^{(l)} = \frac{l!}{2\pi i}\int\limits_{\partial D_n}\frac{W_j'(\xi)/W_j(\xi)}{(\xi-\xi_n)^{l+1}}\,d\xi,$$

where ∂D_n is the boundary of D_n, l is some integer, we have by (3.33),

$$\left|\frac{W_j^{(m)}(\xi_n)}{W_j(\xi_n)}\right| \leqslant |\xi_n|^M,$$

where M is some constant, $m = 1,\dots,k$, $j = 1,\dots,k$.

Because

$$\frac{f_j'}{f_j} = \frac{1}{q}\left(c_j + \xi\frac{W_j'}{W_j} + \xi Q_j'\right),$$

we use induction, and can get

$$\begin{aligned}\frac{f_j^{(p)}}{f_j} = \frac{1}{q^p}\bigg\{&\left(\xi Q_j'\right)^p + p\left(\xi Q_j'\right)^{p-1}\left(\xi\frac{W_j'}{W_j}\right) + \left(\xi Q_j'\right)^{p-1}\left[pc_j + \frac{p(p-1)}{2}\right] \\ &+ \frac{p(p-1)}{2}\left(\xi Q_j'\right)^{p-1}\left(\xi\frac{Q_j''}{Q_j'}\right)\bigg\} + \sum_{l=0}^{p-2}F_l^{(j)}(\xi)\left(\xi Q_j'\right)^l,\end{aligned}$$

where $j = 1,\dots,k$; $p = 2,\dots,k$; $F_l^{(j)}(\xi)$ $(l = 0,\dots,p-2)$ are the polynomials in $\xi^m\frac{W_j^{(m)}}{W_j}$ and $\xi^{m-1}\frac{Q_j^{(m)}}{Q_j'}$ $(m = 2,\dots,p)$ with constant coefficients.

Substituting $f_j(z) = \xi^{c_j}W_j(\xi)e^{Q_j(\xi)}$ into (1.3), by (3.15), (3.30), (3.31), (3.33), on the point ξ_n, for sufficiently large n, we have

$$\begin{aligned}&\frac{1}{q^k}\bigg\{\left(c_{j,n}qD^{1/k}\right)^k + k\left(c_{j,n}qD^{1/k}\right)^{k-1}\xi_n\frac{W_j'}{W_j} \\ &\quad + \left(c_{j,n}qD^{1/k}\right)^{k-1}\left[kc_j + \frac{k(k-1)}{2}\right] + \frac{k(k-1)}{2}\left(c_{j,n}qD^{1/k}\right)^{k-1}\left[\frac{\xi_nD^{1/k-1}D'}{kD^{1/k}} - 1\right]\bigg\} \\ &\quad + O(\rho_n)\left(c_{j,n}qD^{1/k}\right)^{k-2} + \sum_{s=0}^{k-2}P_{1s}(\xi_n)\left(qc_{j,n}D^{1/k}\right)^s + \left[G_0\left(\xi_n^q\right) + g_0\left(\frac{1}{\xi_n^q}\right)\right] = 0,\end{aligned}$$

where P_{1s} are the polynomials in $\frac{W_j^{(m)}}{W_j}$, $\frac{Q_j^{(m)}}{Q_j}$, $\frac{1}{\xi^s}$ $(1 \leqslant s \leqslant k-1,\ 1 \leqslant m \leqslant k)$ and $g_1(\frac{1}{\xi^q}),\dots,g_{k-2}(\frac{1}{\xi^q})$, $G_1(\xi^q),\dots,G_{k-2}(\xi^q)$ with constant coefficients.

Recall that $c_{j,n}^k = -1$, from the above formula, we have

$$\frac{W_j'(\xi_n)}{W_j(\xi_n)} + \frac{k-1}{2k}\frac{D'(\xi_n)}{D(\xi_n)} = O\left(|\xi_n|^{-1}\right). \tag{3.34}$$

Since for $\xi \in D_n$,

$$\Phi(\xi) = E(z) = \xi^{(\sum_{j=1}^{k} c_j)} W_1 W_2 \cdots W_k \exp\left\{\left(\sum_{j=1}^{k} c_{j,n}\right) \int_{\xi_n}^{\xi} \frac{q}{t} D(t)^{1/k}\, dt\right\},$$

from this and (3.34), on the point ξ_n, we have

$$\begin{aligned}\frac{\Phi'(\xi_n)}{\Phi(\xi_n)} &= \left(\sum_{j=1}^{k} c_j\right)\xi_n^{-1} + \sum_{j=1}^{k} \frac{W_j'}{W_j} + \left(\sum_{j=1}^{k} c_{j,n}\right)\frac{q}{\xi_n} D(\xi_n)^{1/k} \\ &= \left(\sum_{j=1}^{k} c_j\right)\xi_n^{-1} + \frac{1-k}{2}\frac{D'(\xi_n)}{D(\xi_n)} + \left(\sum_{j=1}^{k} c_{j,n}\right)\frac{q}{\xi_n} D(\xi_n)^{1/k} + O\left(|\xi_n|^{-1}\right).\end{aligned}$$

On the other hand, from (3.3), we have

$$\frac{\Phi'(\xi)}{\Phi(\xi)} = \left(\sum_{1}^{k} c_j\right)\xi^{-1} + \frac{H'}{H} + h'.$$

Thus on the point ξ_n for n sufficiently large,

$$\frac{H'}{H} + h' = \frac{1-k}{2}\frac{D'(\xi_n)}{D(\xi_n)} + \left(\sum_{j=1}^{k} c_{j,n}\right)\frac{q}{\xi_n} D(\xi_n)^{1/k} + O\left(|\xi_n|^{-1}\right). \tag{3.35}$$

From (3.11) and (3.22), $\sum_{j=1}^{k} c_{j,n} = 0$ must hold. Thus by (3.8) and (3.35), on the point ξ_n, we have

$$\xi_n\left(\frac{H'}{H} + h'\right) = \frac{1-k}{2}\nu\left(|\xi_n|\right)\left(1 + o(1)\right) + O(1). \tag{3.36}$$

If $m \leqslant \tau + \varepsilon$, from (3.5), (3.11), (3.13), we know (3.36) cannot hold. So $m > \tau + \varepsilon$, and from (3.13) and (3.36), we have

$$\sigma(1-\varepsilon) < m < \sigma(1+\varepsilon),$$

since ε is arbitrary, we get $\sigma = m$, which gives $\sigma(G_0(\zeta)) = m/q$ from (3.4). This completes the proof of Theorem 1.5. □

4. Proof of Theorem 1.6

Proof. Suppose that $\lambda_e(f_1 \cdots f_k) < \sigma(G_0)$ $(< \infty)$, since $f_j(z)$ and $f_j(z + 2\pi i)$ $(j = 1, \ldots, k)$ are linearly dependent, from Theorem 1.5, we have $q = 1$, and therefore

$$\sigma(G_0) = m \tag{4.1}$$

holds, where m is some integer. But (4.1) contradicts our assumption. This completes the proof of Theorem 1.6. □

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